Zero-threshold resolvent asymptotics of three-body Schrödinger operators

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ZERO-THRESHOLD RESOLVENT ASYMPTOTICS OF THREE-BODY SCHRÖDINGER OPERATORS

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Abstract. We analyze the spectral properties for three-body Schrödinger operators at the threshold zero and give some results on the asymptotics of resolvent under the condition that zero is a regular point for all two-body Subhamiltonians.

1. Introduction

The Efimov effect theoretically discovered by V. Efimov in 1970 ([5]) describes an interesting and unexpected phenomenon for three-body Schrödinger operators which can be roughly stated as follows. When the essential spectrum of the three-particle Hamiltonian is the positive real axis, and when at least two of its two-body Subhamiltonians have a resonance at the threshold zero, the discrete spectrum of the three-body Schrödinger operator is infinite, even if the interactions are very short-range. This phenomenon is striking if one compares it with the results on the finiteness of eigenvalues of two-body Schrödinger operators or N-body Schrödinger operators whose bottom of essential spectrum is only reached by the spectrum of two-cluster Subhamiltonians ([6, 17]). Since then, many works, both in mathematical and physical literature, are devoted to this subject (see, for example, [1, 2, 3, 9, 10, 13, 15, 16, 19, 21, 23]).

Mathematical study of the three-body Efimov effect for Schrödinger operators is carried out in [13, 15, 23]. The analysis is based on threshold spectral properties of two-body Schrödinger operators in presence of zero resonance which is an interesting topic in itself. See [4, 7, 11] and the references quoted therein and [12, 20, 22] for some more recent results when the potential has a critical decay like $-\frac{\gamma}{|x|^2}$ at the infinity. For N-body systems with $N \geq 4$, R. D. Amado and F. C. Greenwood ([2]) discussed the contribution of zero energy resonance of $(N-1)$-particle subsystems to the discrete spectrum of the total system and argued heuristically that singularities in some integral are not strong enough to make it divergent and concluded that there is no Efimov effect for four or more particle systems. Notice that for N-body systems, there are many possible spectral configurations. In [19], the author of the present work proved the existence of N-body Efimov effect when the bottom of the essential spectrum is strictly negative.

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More precisely, let P be an N-body Schrödinger operator, $N \geq 4$, obtained by removing mass-center from the operator

$$-\sum_{j=1}^{N} \frac{1}{2m_j} \Delta_{x_j} + \sum_{1 \leq i < j \leq N} V_{ij}(x_i - x_j), \quad x_j \in \mathbb{R}^3, \tag{1.1}$$

where x_j and m_j denote the position and mass of the j-th particle. Assume that

$$|V_{ij}(y)| \leq C \langle y \rangle^{-\rho}, \quad y \in \mathbb{R}^3, \rho > 2, \tag{1.2}$$

where $\langle y \rangle = (1 + |y|^2)^{1/2}$. For a cluster decomposition a of the N-body system, let P^a denote the Subhamiltonian associated with a and $\#a$ the number of clusters in a. Assume that the bottom of the essential spectrum, E_0, of P is attained by a three-cluster decomposition $b = (b_1, b_2, b_3)$:

$$\inf \sigma(P^b) = E_0 := \inf \sigma_{ess}(P). \tag{1.3}$$

Let $a_k = (b_i \cup b_j, b_k)$, $k = 1, 2, 3$, be two-cluster decompositions resulting from b, where i, j and k take distinct values in $\{1, 2, 3\}$. By the HVZ Theorem, one has $\sigma(P^{a_k}) = [E_0, \infty[$, $k = 1, 2, 3$. We say that E_0 is *unique three-cluster* if

$$\inf \sigma(P^a) = E_0, a \in \{b, a_1, a_2, a_3\}; \quad \inf \sigma(P^a) > E_0, a \notin \{b, a_1, a_2, a_3\}, \#a \geq 2. \tag{1.4}$$

For technical reasons, we also assume that the inter-cluster interactions related to b are attractive: if the pair (ij) is not included in one cluster of b, then

$$V_{ij}(y) \leq 0. \tag{1.5}$$

Theorem 1.1 ([19]). *Let $N \geq 4$. Let the conditions (1.2) with $\rho > 2$, (1.4) and (1.5) be satisfied. Assume that each of the two-cluster Subhamiltonians P^{a_j}, $j = 1, 2, 3$, has a threshold resonance at E_0. Let $N(\lambda)$ denote the number of the eigenvalues of P below $\lambda < E_0$. Then, there exists $C_0 > 0$ depending only on the reduced masses of the clusters b_1, b_2, b_3 such that*

$$N(\lambda) \geq C_0 |\log(E_0 - \lambda)| \tag{1.6}$$

for $\lambda < E_0$ and λ near E_0.

In fact, Theorem 1.1 is proved in [19] for generalized N-body Schrödinger operators including atomic-type ones and mild local singularities are allowed. The accumulation towards E_0 of eigenvalues of P is clearly due to the presence of threshold resonances of two-cluster Subhamiltonians, which is a phenomenon similar to the well-known three-body Efimov effect. The proof is based on the result of [18] on the spectral analysis of N-body Schrödinger operator near a two-cluster threshold, in particular, the contribution of two-cluster threshold resonance to the singularity of the resolvent. In [21], we studied the existence of two-cluster threshold resonances, which allows us to construct a concrete example of the four-body Efimov effect: there exists some constants $\alpha, \beta, g_j > 0$ such that all conditions (hence the conclusion) of Theorem 1.1 hold for the four-body Schrödinger operator with Yukawa-type potentials

$$\begin{aligned} P &= -\beta(\Delta_{x_1}+\Delta_{x_2}+\Delta_{x_3}) - \frac{\alpha e^{-\alpha|x_1|}}{|x_1|} - g_1\frac{e^{-|x_2-x_3|}}{|x_2-x_3|} \\ &\quad -g_2\Big(\frac{e^{-|x_2|}}{|x_2|}+\frac{e^{-|x_1-x_2|}}{|x_1-x_2|}\Big) - g_3\Big(\frac{e^{-|x_3|}}{|x_3|}+\frac{e^{-|x_1-x_3|}}{|x_1-x_3|}\Big), \quad x_j\in\mathbb{R}^3. \end{aligned} \tag{1.7}$$

The parameters α, β are adjusted to guarantee the unique three-cluster condition with the three-cluster decomposition $b = \{(01), 2, 3\}$ (the mass center being fixed at 0 here), while the coupling constants g_j are varied in such a way that each two-cluster Subhamiltonian has a threshold resonance at some critical value. See [21] for more information. Thus mathematically the N-body Efimov effect exists if the bottom of the essential spectrum is unique three-cluster. Note that our results are not in discrepancy with the claim of [2], because it concerns a different spectral configuration: zero-threshold in four-body problems does not verify the unique three-cluster assumption. It is an interesting open question to show rigorously whether or not the Efimov effect can happen at zero-threshold of four-body systems.

To study the Efimov effect for N-body Schrödinger operators when the unique three-cluster assumption is not satisfied, the first step is to understand the singularities at the threshold of the resolvent of three-cluster Subhamiltonians. In this work, we analyze the threshold zero for three-body Schrödinger operators under the assumption that the potentials are weak enough so that none of the two-body Subhamiltonians has negative eigenvalues, nor zero resonance. It reveals that the related mathematical questions are highly nontrivial. We discuss the notion of three-body zero resonance and give the asymptotics of the resolvent when zero is a regular point. We also give a brief discuss on some open questions on this topic.

Notation. For $s \in \mathbb{R}$ and $k \in \mathbb{Z}$, we denote by $L^{2,s}$ and $H^{k,s}$ the weighted-L^2 and weighted-Sobolev spaces $L^2(\langle x\rangle^{2s}dx)$ and $H^k(\langle x\rangle^{2s}dx)$, respectively. For two Banach spaces B and B', $\mathcal{L}(B, B')$ is the space of linear continuous operators from B to B' and $\mathcal{L}(B) = \mathcal{L}(B, B)$. Set $\mathcal{L}(k, s; k', s') = \mathcal{L}(H^{k,s}, H^{k',s'})$ and $\mathcal{L}(s; s') = \mathcal{L}(0, s; 0, s')$.

2. Zero-resonant states for three-body systems

Let P be a three-body Schrödinger operator obtained from

$$-\sum_{j=1}^{3}\frac{1}{2m_j}\Delta_{x^{(j)}} + \sum_{1\le i<j\le 3} V_{ij}(x^{(i)}-x^{(j)}) \tag{2.1}$$

by removal of the mass center, where $\Delta_{x^{(j)}}$ is the Laplacian in $x^{(j)} \in \mathbb{R}^3$, $m_j > 0$, and $V_{ij}(y)$ satisfies

$$|V_{ij}(y)| \le C\langle y\rangle^{-\rho}, \quad y \in \mathbb{R}^3. \tag{2.2}$$

for some $\rho > 0$. The configuration space is

$$X = \{(x^{(1)}, x^{(2)}, x^{(3)}); \frac{1}{M}\sum_{j=1}^{3} m_j x^{(j)} = 0\},$$

with $M = m_1 + m_2 + m_3$, which is identified with $\mathbb{R}^6$. For a cluster-decomposition $a = \{(ij), k\}$ of the whole system labelled by $\{1, 2, 3\}$, where i, j, k take distinct values in $\{1, 2, 3\}$, denote by (x^a, x_a) the associated clustered Jacobi coordinates in X, $V_a(x^a)$ is the function $V_{ij}(x^{(i)} - x^{(j)})$ expressed in terms of the intracluster coordinates x^a. The Two-body Subhamiltonian associated with a is

$$P^a = -\frac{1}{2\mu_k}\Delta_{x^a} + V_a(x^a), \tag{2.3}$$

μ_a being the effective mass defined by

$$\frac{1}{\mu_a} = \frac{1}{m_i} + \frac{1}{m_j}.$$

Let $\mathcal{A}$ denote the set of two-cluster decompositions for the three-body system. Denote by Δ the Laplacian on $\mathbf{X}$ equipped with the metric $q(x) = \sum_j 2m_j|x_j|^2$ on $\mathbf{X}$. Then P can be written as

$$P = -\Delta + \sum_{a\in\mathcal{A}} V_a(x^a) \tag{2.4}$$

To be simple, we assume in the following that all constants are appropriately normalized. Denote $P_0 = -\Delta$, $P_a = -\Delta + V_a(x^a)$. P_a can also be written as $P_a = -\Delta_{x_a} + P^a$. Let $R(z) = (P - z)^{-1}$, (resp., $R_a(z)$, $R_0(z)$) the resolvent of P (resp., of P_a and P_0) for $z \notin \sigma(P)$. One has the resolvent equation

$$\begin{aligned} R(z) &= R_0(z) - \sum_{a\in\mathcal{A}} R_0(z)V_aR(z) \qquad (2.5)\\ &= R_0(z) - \sum_{a\in\mathcal{A}} R_0(z)V_aR_a(z) + \sum_{a,b\in\mathcal{A},a\neq b} R_0(z)V_aR_a(z)V_bR(z) \end{aligned}$$

It follows that

$$R(z) = (1 - K(z))^{-1}R_0(z)(1 - \sum_{a\in\mathcal{A}} V_aR_a(z)), \tag{2.6}$$

where $K(z) = \sum_{a,b\in\mathcal{A},a\neq b} R_0(z)V_aR_a(z)V_b$. The main assumptions of this work are the following

A. $|V_a(y)| \leq C\langle y\rangle^{-\rho}$, $y \in \mathbb{R}^3$, $\rho > 0$.

B. $P^a \geq 0$ and zero is a regular point (*i.e.*, zero is neither eigenvalue nor resonance) of P^a for any two-cluster decomposition $a = \{(ij), k\}$.

P_0 is a six dimensional Laplacian whose Green function can be explicitly computed. One has the following resolvent asymptotics:

$$R_0(z) = \sum_{j=0}^{N} z^jF_j + \sum_{j=2}^{N} z^j \ln zG_j + O(|z|^{N+\epsilon}), \text{ in } \mathcal{L}(-1, s; 1, -s), s > 2N+1, \tag{2.7}$$

for z near 0 and $z \notin \mathbb{R}_+$. Here $F_j, G_j \in \mathcal{L}(-1, s; 1, -s)$, $s > 2j+1$. More precisely, $F_0 \in \mathcal{L}(-1, s; 1, -s')$ if $s, s' > 1/2$ and $s + s' > 2$ and G_2 is an integral operator with the constant integral kernel

$$G_2(x, y) = -\frac{1}{4|\mathbb{S}^5|}. \tag{2.8}$$

See, for example, [8] or Theorem 2.2 in [22] for Laplacian on a cone. Making use of a partial Fourier transform and the resolvent properties of three-dimensional Laplacian, one sees that

$$\langle u\rangle^{-s} F_0 \langle u\rangle^{-s'} \in \mathcal{L}(L^2(\mathbb{R}^6)), \quad u = x^a \text{ or } x_a \tag{2.9}$$

if $s, s' > 1/2$ and $s + s' > 2$.

Lemma 2.1. *Let $f(x)$, $g(x)$ be locally bounded on $\mathbb{R}^6$ satisfying for some $s, s' < 1$ and $s + s' < -3$, $|f(x)| = O(\langle x\rangle^s)$ and $|g(x)| = O(\langle x\rangle^{s'})$. Then the operator $f(x)F_0 g(x)$ is compact in $L^2(\mathbb{R}^6)$. In particular, F_0 maps $L^{2,s}$ into L^2 if $s > 3$.*

Proof. Let χ be a cut-off around $0 \in \mathbb{R}$: $\chi(\lambda) = 0$ for $|\lambda| \geq 1$; 0 for $|\lambda| \leq 1/2$. Clearly, $f(x)(1 - \chi(P_0))F_0 g(x)$ is a compact operator. The integral kernel of $\chi(P_0)F_0$ is locally bounded and behaves like $O(\frac{1}{|x-y|^4})$ at infinity. Under the condition on s and s', one can check that the integral kernel of $f(x)\chi(P_0)F_0 g(x)$ belongs to $L^2(\mathbb{R}^6 \times \mathbb{R}^6)$. Therefore $f(x)\chi(P_0)F_0 g(x)$ is a Hilbert-Schmidt operator, hence also compact. □

Lemma 2.1 shows in particular that for a two-body Schödinger operator $-\Delta + U(x)$ on $\mathbb{R}^6$ with a sufficiently rapidly decreasing potential $U(x)$, if $u \in L^2(\mathbb{R}^6, \langle x\rangle^{-2s}dx)$ for some $s > 1/2$ and $(-\Delta + U(x))u = 0$, then $u = -F_0 U u$ and u is in L^2. This implies that 0 is never a resonance for six-dimensional two-body Schrödinger operators with rapidly decreasing potentials with $\rho > 2$ sufficiently large. See [8]. Notice however that zero can still be a resonance in any space dimension if $U(x)$ has a critical-decay like $-\frac{\gamma}{|x|^2}$ ([22]). Lemma 2.1 does not directly apply to three-body operators, because the total potential does not decay on the whole configuration space. In the following, we analyze the threshold properties by taking into account the geometry of three-body problems.

Lemma 2.2. *Let a, b be two cluster decompositions with $a \neq b$. Assume $\rho > 7/2$. For $s \in]\frac{1}{2}, \min\{1, \rho - 3\}[$, $\langle x^a\rangle^{-s} F_0 V_a F_0 V_b \langle x\rangle^s$ is a compact operator on $L^2(\mathbb{R}^6)$.*

Proof. Write

$$\langle x^a\rangle^{-s} F_0 V_a F_0 V_b \langle x\rangle^s = (\langle x^a\rangle^{-s} F_0 \langle x^a\rangle^{-s'})(\langle x^a\rangle^{s'} V_a F_0 V_b \langle x\rangle^s).$$

For any $s, s' > 1/2$ and $s + s' > 2$, $\langle x^a\rangle^{-s} F_0 \langle x^a\rangle^{-s'}$ is bounded. Let $a \neq b$. Then (x^a, x^b) forms a coordinate system of $\mathbb{R}^6$ and

$$\langle x\rangle^s \leq C_s(\langle x^a\rangle^s + \langle x^b\rangle^s). \tag{2.10}$$

To show that $\langle x^a\rangle^{-\rho+s'} F_0 \langle x^b\rangle^{-\rho} \langle x\rangle^s$ is compact on L^2, if $s < 1$ and $\rho - s - s' > 3/2$, we use the cut-off in energies introduced in Lemma 2.1. Operator

$$\langle x^a\rangle^{-\rho+s'} \chi_0(P_0) F_0 \langle x^b\rangle^{-\rho} \langle x\rangle^s$$

is compact on L^2 if $\rho - s - s' > 0$. The integral kernel of $\langle x^a\rangle^{-\rho+s'}\chi_0(P_0)F_0\langle x^b\rangle^{-\rho}\langle x\rangle^s$ can be bounded by $\langle x^a\rangle^{-\rho+s'}O(|x-y|^{-4})\langle y^b\rangle^{-\rho}\langle y\rangle^s$ for $|x|, |y|$ and $|x-y|$ large. Since $a \neq b$, one can use (x^a, x^b) as coordinate system on $\mathbb{R}^6$ to evaluate the integral. By (2.10), one has for $\rho > 7/2$

$$
\begin{aligned}
&|\int_{\mathbb{R}^6_y} \frac{1}{\langle x-y\rangle^8}\langle y^b\rangle^{-2\rho}\langle y\rangle^{2s} dy^a\, dy^b| \\
&\leq \; C_\epsilon(\langle x^a\rangle^{2s}\langle x^b\rangle^{-5+\epsilon} + \langle x^b\rangle^{-5+2s+\epsilon}) \qquad (2.11)
\end{aligned}
$$

for any $\epsilon > 0$. This shows that

$$\langle x^a\rangle^{-\rho+s'}O(|x-y|^{-4})\langle y^b\rangle^{-\rho}\langle y\rangle^s$$

is square-integrable on $\mathbb{R}^{12}$ if $s < 1$ and $\rho - s' - s > 3/2$ and it defines Hilbert-Schmidt operator on L^2. If $\rho > \frac{7}{2}$ and $s \in]\frac{1}{2}, \rho - 3[$, there exists $s' > \frac{1}{2}$ such that $2 - s < s' < \rho - s - \frac{3}{2}$. The decay conditions on potentials gives the desired result. □

Proposition 2.3. *Let $a, b \in \mathcal{A}$ with $a \neq b$. Assume that 0 is a regular point of P^a. Then for $\rho > 2$, the limit*

$$R_a(0) = \lim_{z\to 0, z\notin \mathbb{R}_+} R_a(z) \qquad (2.12)$$

exists as operator from $L^2(\langle x^a\rangle^{2s}dx)$ to $L^2(\langle x^a\rangle^{-2s'}dx)$, $s, s' > 1/2$ and $s+s' > 2$. In addition, for $\rho > 7/2$, $s \in]1/2, \min\{1, \rho-3\}[$, $F_0V_aR_a(0)V_b$ is a compact operator on $L^{2,-s}$.

Proof. By the partial Fourier transform in x_a-variables, $P_a = P_0 + V^a(x^a)$ is unitarily equivalent with $P^a + |\xi_a|^2$, ξ_a being the dual variables of x_a. Using the well-known results for two-body Schrödinger operators in dimension three([7]), one sees that

$$\lim_{z\to 0, z\notin\mathbb{R}_+} R_0(z)V_a = F_0V_a$$

in $\mathcal{L}(L^2(\langle x^a\rangle^{-2s}dx))$ for any $s > 1/2$ if $\rho > 2$ and that if 0 is a regular point of P^a, the limit

$$R_a(0) := \lim_{z\to 0, z\notin\mathbb{R}_+} R_a(z) = (1+F_0V_a)^{-1}F_0 : L^2(\langle x^a\rangle^{2s}dx) \to L^2(\langle x^a\rangle^{-2s'}dx) \qquad (2.13)$$

exists and is continuous if $s, s' > 1/2$ and $s + s' > 2$.

To see the second part of Proposition 2.3, we write:

$$F_0V_aR_a(0)V_b = (1 - F_0V_a(1+F_0V_a)^{-1})F_0V_aF_0V_b$$

According to Lemma 2.2, $\langle x^a\rangle^{-s}F_0V_aF_0V_b\langle x\rangle^s$ is a compact on L^2 when $a \neq b$. Since zero is a regular point of P^a and $\rho > 7/2$, $F_0V_a(1+F_0V_a)^{-1}$ is bounded on $L^2(\langle x^a\rangle^{-2s}dx)$. Therefore $F_0V_aR_a(0)V_b$ is compact on $L^{2,-s}$ if $s > 1/2$. □

From Proposition 2.3, one deduces the following

Corollary 2.4. *Assume that* 0 *is a regular point of all two-body subhamiltonians and* $\rho > \frac{7}{2}$. *Then, operator*

$$K = \sum_{a,b\in\mathcal{A},a\neq b} F_0 V_a R_a(0) V_b : L^{2,-s} \to L^{2,-s} \tag{2.14}$$

is compact if $s \in]\frac{1}{2}, \min\{1, \rho - 3\}[$.

By studying the dependence on z of $R_0(z)$ and $R_a(z)$, one can deduce that

$$\lim_{z\to 0,\Re z<0} K(z) = K \tag{2.15}$$

in norm of operators on $L^{2,-s}$. See the next Section.

Corollary 2.5. *Assume the conditions of Corollary 2.4. Then, one has*

$$\ker(1-K) = \{u \in H^{1,-s}; v := Pu \in H^{-1,s} \text{ and } v = \sum_{a\in\mathcal{A}} V_a R_a(0) v\}. \tag{2.16}$$

The null space of P *in* $H^{1,-s}$ *is included in* $\ker(1-K)$. *In particular, if zero is an eigenvalue of* P, *its multiplicity is finite.*

Corollary 2.5 can be deduced from the equation

$$1 - K(z) = R_0(z)(1 - \sum_{a\in\mathcal{A}} V_a R_a(z))(P - z) \tag{2.17}$$

by studying the limit $z \to 0$. The finiteness of the multiplicity of zero-eigenvalue of P follows from the compactness of K. Remark that different from two-cluster threshold problems ([18]), one can not affirm here whether $\ker(1-K)$ coincides with the null space of P in $H^{1,-s}$. Zero is said to be a resonance of P if $\ker(1-K)$ does not coincide with the zero-eigenspace of P. Any nonzero function $u \in \ker(1-K)$ which is not an eigenfunction of P is called a zero-resonant state of P. We say that zero is a regular point of P if $\ker(1-K) = \{0\}$.

3. Resolvent asymptotics at zero-threshold

In this Section, we give some results on the asymptotics of the resolvent at the threshold zero and sketch the proof. To be simple, we only study the resolvent $R(z)$ in the limit $z \to 0$ with $z = -\lambda$, $\lambda > 0$. The results still hold if z approches to zero in a sector $|\arg z - \pi| \le \frac{\pi}{2} - \epsilon$, $\epsilon > 0$. It is known that the assumptions on two-body Subhamiltonians ensures that P has only a finite number of negative eigenvalues ([15]) and $R(-\lambda)$ is well-defined for $\lambda > 0$ small enough.

Theorem 3.1. *Assume the conditions A and B with* $\rho > 7/2$. *Suppose that zero is a regular point of* P. *Then for* $s > 3/2$, *one has for some* $\sigma > 0$

$$R(-\lambda) = R(0) + O(|\lambda|^\sigma), \quad \lambda \to 0_+, \tag{3.1}$$

in $\mathcal{L}(-1, s; 1, -s)$. *Here* $R(0)$ *is defined by*

$$R(0) = (1 - K)^{-1} F_0 (1 - \sum_{a \in \mathcal{A}} V_a R_a(0)) \tag{3.2}$$

To begin with, remark that the free-resolvent $R_0(\lambda)$ can be expanded in λ as operator from $L^2(\langle x^a \rangle^{2s} dx)$ to $L^2(\langle x^b \rangle^{-2s} dx)$ for any two-cluster decompositions a and b.

When $a = b$, by the partial Fourier transform in x_a-variables and the Green-function for three-dimensional Laplacian, one has for any $N \in \mathbb{N}$

$$R_0(-\lambda) = F_0 + \sum_{k=1}^{N} r(\lambda)^n H_n + O(|\lambda|^{N/2+\epsilon}) \tag{3.3}$$

in $\mathcal{L}(L^2(\langle x^a \rangle^{2s} dx); L^2(\langle x^a \rangle^{-2s} dx))$ for $s > N + 1$. Here F_0 is given before and $r(\lambda)$ and H_n are defined in terms of (x^a, ξ_a) with ξ_a the dual variables of x_a:

$$r(\lambda) := \frac{\lambda}{\sqrt{|\xi_a| + \lambda} + |\xi_a|}, \tag{3.4}$$

$$H_n := \frac{(-1)^n}{4\pi n!} |x^a - y^a|^{n-1} e^{-|\xi_a||x^a - y^a|}, \quad n \geq 1. \tag{3.5}$$

Remark that

$$r(\lambda)^n H_n = O(\lambda^{n/2})$$

in $\mathcal{L}(L^2(\langle x^a \rangle^{2s} dx); L^2(\langle x^a \rangle^{-2s} dx))$ for $s > n+1$. In particular, one can deduce that

$$R_0(-\lambda) = F_0 + O(|\lambda|^{\epsilon}), \epsilon > 0, \tag{3.6}$$

in $\mathcal{L}(L^2(\langle x^a \rangle^{2s} dx); L^2(\langle x^a \rangle^{-2s'} dx))$ if $s, s' > 1/2$ and $s + s' > 2$.

When $a \neq b$, we use the formula of the distributional kernel $R_0(x, y; -\lambda)$ for $R_0(-\lambda)$

$$R_0(x, y; -\lambda) = \frac{1}{(4\pi)^3 |x - y|^4} \int_0^\infty e^{-\frac{1}{4s} - \lambda s |x-y|^2} s^{-3} \, ds. \tag{3.7}$$

See formula (6.49) in p. 232 of [14]. It follows that

$$R_0(x, y; -\lambda) = \frac{1}{(4\pi)^3 |x - y|^4} (\tau(0) + \lambda |x - y|^2 \int_0^\infty e^{-\lambda s |x-y|^2} \tau(s) \, ds) \tag{3.8}$$

where $\tau'(s) = e^{-\frac{1}{4s}} s^{-3}$ and $\tau(s) = O(s^{-2})$ as $s \to +\infty$. As in Section 3, by separating high and low energy parts and by making use of the bound

$$e^{-\lambda s |x-y|^2} \leq C_{\sigma'} (\lambda s |x - y|^2)^{-\sigma'}, \quad \sigma' \in]\frac{1}{2}, 1[,$$

for $|x - y|$ large, one deduces that if $\alpha > 3/2$, the Hilbert-Schmidt norm of the operator with integral kernel

$$\frac{\lambda}{(4\pi)^3 |x - y|^2} \langle x^a \rangle^{-\alpha} \langle y^b \rangle^{-\alpha} \int_0^\infty e^{-\lambda s |x-y|^2} \tau(s) \, ds$$

is bounded by $O(\lambda^\sigma)$ with $\sigma = 1 - \sigma'$. Therefore, in the case $a \neq b$, one has

$$R_0(-\lambda) = F_0 + O(|\lambda|^\sigma) \tag{3.9}$$

in $\mathcal{L}(L^2(\langle x^b\rangle^{2s}dx); L^2(\langle x^a\rangle^{-2s}dx))$ for $s > 3/2$. In the same way, one can show that the above expansion also holds in $\mathcal{L}(L^2(\langle x^b\rangle^{2s}\langle x^a\rangle^{-2r}dx); L^2(\langle x^a\rangle^{-2s-2r}dx))$ if $r \in]0,1[$, $s > 3/2$ and $\sigma < \frac{1-r}{2}$. Similarly, one can prove the following results for $R_a(z)$.

Proposition 3.2. *Assume the conditions A and B with $\rho > 7/2$. Let a, b be two-cluster decompositions with $a \neq b$. The following expansions hold as $\lambda \to 0_+$.*

(a). For any $s > 3/2$ and $\sigma \in]0, 1/2[$, one has

$$R_a(-\lambda) = R_a(0) + O(\lambda^\sigma), \tag{3.10}$$

in $\mathcal{L}(L^2(\langle x^b\rangle^{2s}dx); L^2(\langle x^a\rangle^{-2s}dx))$.

(b). One has

$$R_0(-\lambda)V_aR_a(-\lambda)V_b = F_0V_aR_a(0)V_b + O(\lambda^\sigma), \tag{3.11}$$

in $\mathcal{L}(1, -s; 1, -s)$ for $s \in]1/2, \min\{1, \rho - 3\}[$ and $\sigma \in]0, \frac{1-s}{2}[$.

Theorem 3.1 follows from Proposition 3.2 and equation (2.6) by noticing that if $s > 3/2$, one has for some $\sigma > 0$

$$R_0(-\lambda)(1 - \sum_a V_aR_a(-\lambda)) = F_0(1 - \sum_{a\in\mathcal{A}} V_aR_a(0)) + O(|\lambda|^\sigma) \tag{3.12}$$

in $\mathcal{L}(-1, s; 1, -s')$ for any $s' > 1/2$ and that $1 - K(-\lambda) = 1 - K + O(\lambda^\sigma)$ in $\mathcal{L}(1, -s'; 1, -s')$ for $s' > 1/2$ and sufficiently close to 1/2.

Remark. Under some additional conditions, we can give a leading term of the resolvent $R(z)$ when $\ker(1-K)$ is non trivial. Let $\dim\ker(1-K) = m \neq 0$ and $\{\varphi_1, \cdots, \varphi_m\}$ a basis of $\ker(1-K)$. Assume that all φ_j are in $L^{2,s}$ for some $s > 0$ sufficiently large. Then one can study the inverse of the following Grushin problem

$$\mathcal{K}(z) = \begin{pmatrix} 1 - K(z) & T \\ T^* & 0 \end{pmatrix} : L^{2,-s} \times \mathbb{C}^m \to L^{2,-s} \times \mathbb{C}^m, \tag{3.13}$$

where $s > 1/2$, T is defined by

$$Tc = \sum_{j=1}^m c_j\varphi_j, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_\mu \end{pmatrix} \in \mathbb{C}^m$$

and T^* is the formal adjoint of T. The inverse of $\mathcal{K}(z)$ can be explicitly calculated as in [18]. Set $\mathcal{K}(z)^{-1}$ into the form

$$\mathcal{K}(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

Then one has a representation formula for $(1-K(z))^{-1}$ when z is away from the positive real axis:

$$(1 - K(z))^{-1} = E(z) - E_+(z)(E_{-+}(z))^{-1}E_-(z). \tag{3.14}$$

The leading terms of $E(z)$ and $E_\pm(z)$ can be computed in the case where zero is a regular point of P. The calculation of singularities of $(1-K(z))^{-1}$ is then reduced to the analysis of the $m \times m$ matrices $E_{-+}(z)$. When all φ_j decays sufficiently rapidly, one can establish an asymptotics of the form

$$E_{-+}(z) = zE_{-+}^{(1)} + o(|z|) \tag{3.15}$$

when $z \to 0$ in a small sector around the negative real axis. If the matrix $E_{-+}^{(1)}$ is invertible, then one can show that the leading singularity of the resolvent $R(z)$ at zero is of the form $\frac{\Pi_m}{z}$ where Π_m is an operator of rank m. But it is known that threshold eigenfunctions do not decay rapidly in general and we are unable to produce an example such that these conditions are satisfied.

Open Questions. As the reader may feel, many questions on the zero threshold of three-body Schrödinger operators remain open. Here we discuss some of them.

(1) Can one establish a higher order expansion for $R(z)$ when 0 is a regular point of P under stronger decay assumptions on V_a? As seen above, the difficulty comes from the validity of the expansion for $V_a R_0(z) V_b$ with $a \neq b$. Other more refined methods, such as microlocal analysis, may be useful.

(2) Does there exist zero resonant states for three-body operators, or more generally, does $\ker(1-K)$ coincide with the null space of P in $L^{2,-s}$?

(3) What is the leading term of the resolvent $R(z)$ when $z \to 0$ when 0 is not a regular point of P? If $\ker(1-K)$ coincides with the zero-eigenspace of P, one expects that the leading singularity is of the form $\frac{1}{z}$. Even this result is not easy to prove without additional decay assumptions on zero-eigenfunctions. A more important question is if zero-resonant states do exist, what are their contributions to the singularities of $R(z)$? The answer to these questions is crucial to see if three-body zero-eigenfunctions or zero-resonant states can produce a four-body Efimov effect.

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